PMM U.S.S.R.,Vol.46,pp.66-74

0021-8928/83/1 0066 $7.50/0

UDC 532.529

ON THE HYPERBOLICITY, STABILITY AND CORRECTNESS OF THE CAUCHY PROBLEM FOR THE SYSTEM OF EQUATIONS OF TWO-SPEED MOTION OF TWO-PHASE MEDIA*

L.A. KLEBANOV, A.E. KROSHILIN, B.I. NIGMATULIN, and R.I. NIGMATULIN

Unsteady one-dimensional two-speed flows of a disperse stream are investigated and properties of the respective system of differential equations are investigated. Propagation of small perturbations is studied on the example of a mixture of barotropic gas with incompressible particles. It is pointed out that the nonhyperbolicity and instability of small perturbations peculiar to the system of differential equations are due to incomplete definition of interaction between phases and inside the dispersed phase, and to transport effects, and unrelated to acoustic perturbation propagation. Estimates are obtained for the characteristic times of instability development in streams of the drop and bubble structure.

The "conditional" correctness of the Cauchy problem for the nonhyperbolic system of differential equations considered here is established on the example of the problem of uniform motion of a disperse mixture of incompressible phases, when the dispersed particles are gradually drawn into the carrier phase motion and the slipping velocity approaches zero. The inclusion internal pressure effect on flow stability is studied.

Many problems of hydrodynamics of heterogeneous media with unequal phase velocities differ from the respective problems of hydrodynamics of single-phase fluid /1/. A closed system of differential equations was proposed in the fundamental paper /2/ for the definition of two-speed two-phase flows, and its nonhyperbolicity in the case of incompressible phases noted. It was shown in /3/ that the more general case of compressible phase flows the system of differential equations is nonhyperboic for real values of slipping (velocity difference). The method of averaging was used in /4,5/ for deriving equations of flow for a perfect fluid with small particles, and the nonhyperbolicity of the system of equations with the consequent flow instability pointed out. Layered ocean flows were considered in /6/, and the appearance of nonhyperbolicity in the considered case noted. The complexity of numerical calculations of two-speed flows is described in detail in the survey paper /7/.

1. The system of differential equations that defines the unsteady flow of a one-dimensional two-phase two-speed stream of a monodisperse mixture with barotropic phases but without phase transition and collision effects is of the form /1/

$$\frac{\partial(\rho_1{}^\circ\alpha_1)}{\partial t}+\frac{\partial(\rho_1{}^\circ\alpha_1v_1)}{\partial x}=0 \tag{1.1}$$

$$\frac{\partial(\rho_2{}^\circ\alpha_2)}{\partial t}+\frac{\partial(\rho_2{}^\circ\alpha_2v_2)}{\partial x}=0,\quad \alpha_1+\alpha_2=1$$

$$\rho_1{}^\circ\alpha_1\frac{d_1v_1}{dt}=-\alpha_1\frac{\partial p}{\partial x}-F+\rho_1{}^\circ\alpha_1g$$

$$\rho_2{}^\circ\alpha_2\frac{d_2v_2}{dt}=-\alpha_2\frac{\partial p}{\partial x}+F+\rho_2{}^\circ\alpha_2g,$$

$$\frac{d_i}{dt}=\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x},\quad \rho_i{}^\circ=\rho_i{}^\circ(p),\quad i=1,2$$

where α_i is the volume concentration, ρ_i° is the true density, p is the pressure, v_i is the velocity, F is the phase interaction force due to the difference of phase velocities per unit of mixture volume, and g is the external mass forces intensity. Subscripts $i=1,2$ relate, respectively, to the first and second phase. We shall consider the carrier phase as the first $(i=1)$ and the dispersed one $(i=2)$ as the second.

Force F of phase interaction is represented in the form of the sum of several components, such as viscous friction force F_μ, joined mass forces F_m, forces F_b due to unsteadiness of the viscous boundary layer (Basset forces), etc.

$$F=F_\mu+F_m+F_b+\ldots \tag{1.2}$$

$$F_\mu=\alpha_1\alpha_2K_\mu(v_1-v_2),\quad F_m=\frac{\rho_1{}^\circ\alpha_1\alpha_2}{2}\left(\frac{d_1v_1}{dt}-\frac{d_2v_2}{dt}\right)$$

*Prikl.Matem.Mekhan.,46,No.1,83-95,1982

At fairly low Reynolds numbers $R_{12} = 2a|v_{12}|\rho_1^\circ\mu_1^{-1}$, where a is the particle radius, μ_1 is the dynamic viscosity coefficient of the carrier phase, and $v_{12} = v_1 - v_2$), we have for K_μ the Stokes law

$$K_\mu = \frac{9\mu}{2a^2}\varphi_f(\alpha_2) = \frac{\rho_2^\circ}{\tau_\mu}, \quad \alpha_2 \to 0, \quad \varphi_f \to 1 \tag{1.3}$$

where φ_f is a coefficient that takes into account the volume content of dispersed particles α_2 on force f per particle, and τ_μ is the relaxation time of equalization of phase velocities.

Let the interphase force be determined only by the friction force ($F = F_\mu$), i.e. depend only on α_2 and $v_{12} = v_1 - v_2$, which is usually the case with gaseous suspensions. (The effect of apparent mass is taken into account in Sect.3). In that case the characteristic directions $dx/dt = \lambda$ of the system of Eqs.(1.1) are roots of the following equation /2,3,7/:

$$\frac{1}{\rho_* c_*^2}(\lambda - v_1)^2(\lambda - v_2)^2 - \frac{\alpha_1}{\rho_2^\circ}(\lambda - v_2)^2 - \frac{\alpha_2}{\rho_1^\circ}(\lambda - v_1)^2 = 0 \tag{1.4}$$

$$\frac{1}{\rho_*} = \frac{\alpha_1}{\rho_1^\circ} + \frac{\alpha_2}{\rho_2^\circ}, \quad \frac{1}{\rho_* c_*^2} = \frac{\alpha_1}{\rho_1^\circ c_1^2} + \frac{\alpha_2}{\rho_2^\circ c_2^2}; \quad c_i^{-2} = \frac{d\rho_i^\circ}{dp}; \quad i = 1, 2$$

where c_1 and c_2 are propagation velocities of acoustic perturbations in the first and second phase, respectively.

Note that the allowance for lack of barotropicity and of temperature nonequilibrium ($T_1 \neq T_2$) of the mixture does not greatly effect the equation of characteristics. System (1.1) is in that case supplemented by equations of conservation of internal energy u_i for each of the mixture constituents. The two characteristic directions of the derived system of equal to the first and second phase velocities (v_1 and v_2) and correspond to transport velocities of the small temperature perturbations. The remaining four eigenvalues are roots of Eq.(1.4), and in this case

$$c_i^{-2} = \frac{\partial\rho_i^\circ}{\partial p}\left[1 - \frac{p}{(\rho_i^\circ)^2}\frac{\partial\rho_i^\circ}{\partial u_i}\right]^{-1}$$

Analysis of Eq.(1.4) shows that two eigenvalues are always real, while the remaining two may be complex. When the slipping velocity v_{12} is considerably lower than the speeds of sound c_1 and c_2, the roots of Eq.(1.4) are of the form

$$\lambda^{(1,2)} = \rho_*\left[\frac{\alpha_1}{\rho_1^\circ}v_2 + \frac{\alpha_2}{\rho_2^\circ}v_1 \pm iv_{12}\left(\frac{\alpha_1\alpha_2}{\rho_1^\circ\rho_2^\circ}\right)^{1/2}\right], \quad i^2 = -1 \tag{1.5}$$

$$\lambda^{(3,4)} = v_* \pm c_*; \quad v_* = \rho_*\left(\frac{\alpha_1}{\rho_1^\circ}v_1 + \frac{\alpha_2}{\rho_2^\circ}v_2\right)$$

which shows that the pair of real values $\lambda^{(3,4)}$ corresponds to propagation of acoustic waves in the medium considered. The quantity c_* corresponds to the propagation velocity of acoustic perturbations when $v_1 = v_2 = 0$. The quantity v_* is an addition to the propagation velocity of small perturbations due to the mixture motion. Note that v_* differs from the mixture means mass velocity v defined as

$$\rho v = \rho_1^\circ\alpha_1 v_1 + \rho_2^\circ\alpha_2 v_2, \quad \rho = \rho_1^\circ\alpha_1 + \rho_2^\circ\alpha_2 \tag{1.6}$$

It follows from (1.5) that the pair of complex eigenvalues is unrelated to acoustic pertrubations, and is solely due to phase transport effects at velocities v_1 and v_2. At points where $v_1 = v_2 = v$ eigenvalues $\lambda^{(1,2)}$ or characteristic directions become real and equal v.

2. Consider one of the steady state solutions of the system of Eqs.(1.1) which yields a nonzero phase slipping v_{12} owing to external mass forces of intensity g constant with respect to time and coordinate.

If both phases are incompressible, we have a steady state (parameters α_1, v_1, v_2, p are time independent), homogeneous (parameters $W = (\alpha_1, v_1, v_2)$ are independent of the coordinate) solution of system (1.1) that satisfies the equations

$$-\alpha_1\frac{\partial p}{\partial x} - F + \rho_1^\circ\alpha_1 g = 0, \quad -\alpha_2\frac{\partial p}{\partial x} + F + \rho_2^\circ\alpha_2 g = 0, \quad \partial p/\partial x = \rho g, \quad F = K_\mu\alpha_1\alpha_2(v_1 - v_2) \tag{2.1}$$

If at least one of the phases is compressible, then owing to pressure variation along x, the density of that phase also varies with x. It can be shown that a steady state homogeneous (with parameters $W = (\alpha_1, v_1, v_2)$ independent of x) solution does not exist. It is possible to visualise a case in which the phase compressibility has only a slight effect at some finite distances L. For this it is necessary that

$$\frac{1}{c_i^2}\frac{\partial p}{\partial x} = \frac{\partial \rho_i^\circ}{\partial x} \ll \frac{\rho_i^\circ}{L}, \quad \frac{\rho g}{c_i^2} \ll \frac{\rho_i^\circ}{L} \tag{2.2}$$

In the case of compressible phases the steady state solution of system (1.1) does not much differ from the steady state solution which follows from (2.1), and is of the form

$$W_0 = (\alpha_1, v_1, v_2) = \text{const}, \ \frac{\partial p}{\partial x} = \rho g, \quad v_{12} = v_1 - v_2 = \frac{(\rho_1^\circ - \rho_2^\circ) g}{K_\mu} \tag{2.3}$$

To simplify exposition we shall consider the case when the dispersed phase substance (particles) is incompressible

$$\rho_2^\circ = \text{const} \quad (c_2 = \infty) \tag{2.4}$$

Consider the development of weak perturbations of solution (2.3)

$$W = W_0 + W' \exp\left[i(kx + \omega t)\right], \quad (W' \ll W_0, \ i^2 = -1) \tag{2.5}$$

The corresponding dispersion equation is of the form

$$\begin{aligned}
&\omega^4 - \omega^3\left\{2v_{12}k + \frac{iK_\mu\rho}{\rho_1^\circ\rho_2^\circ}\right\} + \omega^2\left\{v_{12}^2k^2 - \frac{\rho_1^\circ}{\alpha_1\rho_*}c_1^2k^2 + \right.\\
&\left.\frac{iK_\mu v_{12}\left[(\rho + \varphi\rho_1^\circ)(\rho_1^\circ - \rho_2^\circ) + \rho\rho_1^\circ\right]}{\rho_1^\circ\rho_2^\circ(\rho_1^\circ - \rho_2^\circ)}k\right\} + \omega\left\{2v_{12}c_1^2k^3 + \left[\frac{iK_\mu}{\alpha_1\rho_2^\circ}c_1^2 - \frac{iK_\mu}{\rho_1^\circ}\left(\alpha_2 + \frac{2\rho}{\rho_1^\circ - \rho_2^\circ}\right)v_{12}^2\right]k^2 + \right.\\
&\left.\frac{K_\mu^2 v_{12}}{\rho_2^\circ(\rho_1^\circ - \rho_2^\circ)}k\right\} - v_{12}^2c_1^2k^4 + \left\{\frac{iK_\mu\rho}{\rho_1^\circ(\rho_1^\circ - \rho_2^\circ)}v_{12}^3 - \frac{iK_\mu}{\alpha_1\rho_2^\circ}v_{12}c_1^2(2\alpha_1 + \varphi - \alpha_2)\right\}k^3 - \frac{K_\mu^2\rho v_{12}^2(2\alpha_1 + \varphi - \alpha_2)}{\alpha_1\rho_1^\circ\rho_2^\circ(\rho_1^\circ - \rho_2^\circ)}k^2 = 0
\end{aligned} \tag{2.6}$$

$$\tau_\mu = \alpha_1\alpha_2\rho_2^\circ\left(\frac{\partial F_\mu}{\partial v_{12}}\right)^{-1}, \quad \varphi = \frac{\tau_\mu}{\rho_2^\circ v_{12}}\left(\frac{\partial F_\mu}{\partial \alpha_1}\right), \quad v_{12} = \frac{\rho_1^\circ - \rho_2^\circ}{\rho_2^\circ}g\tau_\mu \tag{2.7}$$

We distinguish two types of solutions.

Solution of the first type with k real. Four complex values of ω can be found for every k in (2.6)

$$k = \text{Re}\{k\}\,(\text{Im}\{k\} = 0); \quad \omega = \text{Re}\{\omega\} + i\,\text{Im}\{\omega\} \tag{2.8}$$

that yield for each k four solutions that correspond to waves (of length λ)

$$W'_{(k)} = A\exp(-D_{(k)}t)\sin\left[\frac{2\pi}{\lambda_{(k)}}(x + c_{(k)}t)\right], \ c_{(k)} = \text{Re}\{\omega(k)\}k^{-1}, \quad D_{(k)} = \text{Im}\{\omega(k)\}, \quad \lambda_{(k)} = 2\pi k^{-1} \tag{2.9}$$

which are sinusoidal with respect to coordinate and increase or are damped with time.

Consider the limit cases of short ($\lambda \to 0$ or $k \to \infty$) and long ($\lambda \to \infty$ or $k \to 0$) waves when $v_{12}/c_1 \ll 1$. Separation of real and imaginary parts as $k \to \infty$ yields

$$\begin{aligned}
&\omega^{(1,2)} = \left(\frac{\rho_1^\circ}{\alpha_1\rho_*}\right)^{1/2}c_1k + i\frac{\alpha_1\alpha_2}{2\tau_\mu}\rho_2^\circ\rho_*\left(\frac{1}{\rho_1^\circ} - \frac{1}{\rho_2^\circ}\right)^2 + O\left(\frac{v_{12}}{c_1}\right)\\
&\omega^{(3,4)} = \frac{\alpha_1\rho_* v_{12}}{\rho_1^\circ}\left[1 \pm i\left(\frac{\alpha_2\rho_1^\circ}{\alpha_1\rho_2^\circ}\right)^{1/2}\right]k + O\left(\frac{v_{12}}{c_1}\right);\\
&O\left(\frac{v_{12}}{c_1}\right) = \text{Re}\left\{O\left(\frac{v_{12}}{c_1}\right)\right\}; \quad \text{Im}\left\{O\left(\frac{v_{12}}{c_1}\right)\right\} = 0
\end{aligned} \tag{2.10}$$

Similarly, as $k \to 0$

$$\begin{aligned}
&\omega^{(1,2)} = \left(\frac{\rho_1^\circ}{\alpha_1\rho}\right)^{1/2}c_1k + i\frac{\tau_\mu}{2}\frac{\alpha_2\rho_1^\circ}{\rho_2^\circ}\left(\frac{\rho_1^\circ - \rho_2^\circ}{\rho}\right)c_1^2k^2\\
&\omega^{(3)} = i\frac{K_\mu\rho}{\rho_1^\circ\rho_2^\circ} + O\left(\frac{v_{12}}{c_1}\right), \quad \omega^{(4)} = (2\alpha_1 + \varphi - \alpha_2)v_{12}k -\\
&\quad i\alpha_1\tau_\mu\left\{(2\alpha_1 + \varphi - \alpha_2 - 1)^2 + \frac{\rho_1^\circ\alpha_2}{\rho_2^\circ\alpha_1}\left(2\alpha_1 + \varphi - \alpha_2\right)^2\right\}v_{12}^2k^2 +\\
&\quad O\left(\frac{v_{12}}{c_1}\right); \quad O\frac{v_{12}}{c_1} = \text{Re}\left\{O\left(\frac{v_{12}}{c_1}\right)\right\}, \quad \text{Im}\left\{O\left(\frac{v_{12}}{c_1}\right)\right\} = 0
\end{aligned} \tag{2.11}$$

In both limit cases the first two roots correspond to acoustic perturbation propagation at frozen c^f or equilibrium c^e speeds of sound and damping coefficients D^f and D^e, respectively

$$\begin{aligned}
&k \to \infty, \ c_{(k)}^{(1,2)} \to \pm c^f = c_1\frac{\rho_1^\circ}{\alpha_1\rho_*}, \quad D_{(k)}^{(1,2)} \to D^f = \frac{\alpha_1\alpha_2}{2\tau_\mu}\rho_2^\circ\rho_*\left(\frac{1}{\rho_1^\circ} - \frac{1}{\rho_2^\circ}\right)^2\\
&k \to 0, \ c_{(k)}^{(1,2)} \to \pm c^e = \pm c_1\frac{\rho_1^\circ}{\alpha_1\rho}, \quad D_{(k)}^{(1,2)} \to D^e = \frac{\tau_\mu}{2}\frac{\alpha_2\rho_1^\circ}{\rho_2^\circ}\left(\frac{\rho_1^\circ - \rho_2^\circ}{\rho}\right)^2c_1^2k^2
\end{aligned} \tag{2.12}$$

Since damping coefficients are positive, consequently in conformity with (2.9) the acoustic perturbations are damped and do not affect stability of the initial steady state.

The third and fourth roots in (2.10) and (2.11) are independent of the speed of sound in the carrier phase, and correspond to propagation of convection perturbations. It will be seen that when $v_{12} \neq 0$ and $\alpha_2 > 0$ the third or fourth root in (2.10) (depending on the sign of v_{12}) as $k \to \infty$, and the fourth root in (2.11) as $k \to 0$ yield negative $D = \operatorname{Im}\{\omega\}$ which corresponds to an exponential increase of perturbations. Thus the system of Eqs.(1.1) admits for long, as well as for short waves increasing convection perturbations, which renders unstable the initial steady state homogeneous solution (2.3) with nonzero phase slipping ($v_{12} \neq 0$) and nonzero concentration ($\alpha_2 > 0$) of the dispersed phase. Note that when $v_{12} = 0$ or $\alpha_2 = 0$ the appropriate convection perturbations becomes neutrally unstable.

The fact that $D = \operatorname{Im}\{\omega\} \to -\infty$ when $v_{12} \neq 0$, $\alpha_2 > 0$ and $k \to \infty$ shows the incorrectness of the Cauchy problem formulation for Eqs.(1.1) with initial data close to the homogeneous steady state (2.3) with nonzero phase slip ($v_{12} \neq 0$) and nonzero dispersed phase content ($\alpha_2 > 0$).

Solutions of the second type with ω real. Here, from (2.6) we obtain for k the complex value

$$\omega = \operatorname{Re}\{\omega\},\ \operatorname{Im}\{\omega\} = 0;\ k = \operatorname{Re}\{k\} + i \operatorname{Im}\{k\} \tag{2.13}$$

This method was used in /8/ for analyzing small perturbations in disperse mixtures in the presence of temperature effects and phase transitions. Solution (2.13) of the second type define periodic perturbations of fixed frequency and amplitude (proper for each point x). The respective waves either increase or are damped on the x axis.

It was shown in /8/ that, when the mixture is initally in the equilibrium state ($v_1 = v_2$), small perturbations that correspond to acoustic wave propagation are stable, as shown above. The equilibrium and frozen speeds of sound and of respective damping coefficients coincide with those calculated above. It can be shown that for finite wave lengths and perturbation frequencies the speed of sound and the damping coefficients which correspond to the two-wave type (i.e. the actual wave length or the actual perturbation frequency) are different.

3. Taking into account that in the considered here limit cases of ($k \to \infty$ and $k \to 0$) the convection wave propagation velocites and the respective damping coefficients are independent of the speed of sound, we shall investigate the effect of phase relative motion in the input (steady) state and of the interphase force due to the additional masses on the stability of convection perturbations and on the related to it lack of hyperbolicity of system (1.1), using a simpler model of the disperse medium with incompressible phases.

The system of Eqs.(1.1) of continuity and momentum with allowance for the interphase force due to the effect of additional masses in the case of a disperse system with incompressible phases is of the form /1/

$$\begin{aligned}
&\frac{\partial \alpha_1}{\partial t} + \frac{\partial(\alpha_1 v_1)}{\partial x} = 0,\ \frac{\partial \alpha_2}{\partial t} + \frac{\partial(\alpha_2 v_2)}{\partial x} = 0,\quad \alpha_1 + \alpha_2 = 1\\
&\rho_1^\circ \alpha_1 \frac{d_1 v_1}{dt} = -\alpha_1 \frac{\partial p}{\partial x} - \frac{\rho_1^\circ \alpha_1 \alpha_2}{2}\left(\frac{d_1 v_1}{dt} - \frac{d_2 v_2}{dt}\right) - F_\mu + \rho_1^\circ \alpha_1 g\\
&\rho_2^\circ \alpha_2 \frac{d_2 v_2}{dt} = -\alpha_2 \frac{\partial p}{\partial x} + \frac{\rho_1^\circ \alpha_1 \alpha_2}{2}\left(\frac{d_1 v_1}{dt} - \frac{d_2 v_2}{dt}\right) + F_\mu + \rho_2^\circ \alpha_2 g\\
&F_\mu = \alpha_1 \alpha_2 K_\mu (v_1 - v_2)
\end{aligned} \tag{3.1}$$

Eliminating the pressure gradient from the equations of momenta, adding the two continuity equations and substituting the new variables w and v_V for v_1 and v_2, and introducing coefficients χ and χ_*

$$w = v_1 - \chi v_2,\qquad v_V = \alpha_1 v_1 + \alpha_2 v_2,\quad \chi = \frac{1}{3}\left(1 + \frac{2\rho_2^\circ}{\rho_1^\circ}\right)\quad \chi_* = \frac{1}{3}\left(1 + \frac{2\rho_2^\circ}{\rho_*}\right) \tag{3.2}$$

we transform the system of Eqs.(3.1) to the form

$$\begin{aligned}
&\frac{\partial w}{\partial t} + V \frac{\partial w}{\partial x} - \left(\frac{\chi}{\chi_*}\right) v_{12}^2 \frac{\partial \alpha_1}{\partial x} = G\\
&\frac{\partial \alpha_1}{\partial t} + V \frac{\partial \alpha_1}{\partial x} + \frac{\alpha_1 \alpha_2}{\chi_*} \frac{\partial w}{\partial x} = 0,\quad \frac{\partial v_V}{\partial x} = 0\\
&\frac{\partial v_V}{\partial t} + \frac{3\chi_*}{\rho + 2\rho_2^\circ} \frac{\partial p}{\partial x} = \frac{2\alpha_1 \alpha_2 v_{12}}{\chi_*} \frac{\partial w}{\partial x} + \frac{(\alpha_2 - \alpha_1 \chi) v_{12}^2}{\chi_*} \frac{\partial \alpha_1}{\partial x} + H\\
&V = \frac{v_V \chi - w(\alpha_1^2 \chi - \alpha_2^2)}{(\alpha_2 + \alpha_1 \chi)\chi_*},\quad G = \frac{2}{3\rho_1^\circ}\left[(\rho_1^\circ - \rho_2^\circ) g - \frac{F_\mu}{\alpha_1 \alpha_2}\right]\\
&H = g + \frac{2(\rho_1^\circ - \rho_2^\circ)}{\rho_1^\circ(\rho + 2\rho_2^\circ)} F_\mu,\quad v_{12} = v_1 - v_2 = \frac{v_V(\chi - 1) + w}{(\chi - 1)\alpha_1 + 1}
\end{aligned} \tag{3.3}$$

Since the first two equations do not contain derivatives of p and v_V, the structure of system of w and α_1 can be investigated independently. The respective characteristic directions of that subsystem are complex

$$\lambda^{(1,2)} = V \pm i v_{12} \left[\left(\frac{\chi}{\chi_*} \right) \frac{\alpha_1 \alpha_2}{\chi_*} \right]^{1/2} \tag{3.4}$$

The comparison of (1.5) with (3.4) show that the characteristics of system (1.1) are not qualitatively altered when the effect of apparent masses is taken into account, and that it continues to have two complex characteristics.

We consider, as in Sect.2, the development of weak perturbations of the type (2.5) of the steady homogeneous state /solution/ (2.3). The third of Eqs.(3.3) implies that the perturbation amplitude v_V is zero, hence perturbations G or $F_\mu/(\alpha_1\alpha_2)$ are defined by perturbations w' and α_1'

$$G' = -f_\alpha \alpha_1' - f_w w' \quad (v_V' = 0) \tag{3.5}$$

$$f_w = \frac{2}{3\rho_1^\circ} \frac{\partial}{\partial w} \left[\frac{F_\mu(\alpha_1, w, v_V)}{\alpha_1 \alpha_2} \right] = \frac{2K_\mu}{3\rho_1^\circ (\chi\alpha_1 + \alpha_2)}$$

$$f_\alpha = \frac{2}{3\rho_1^\circ} \frac{\partial}{\partial \alpha_1} \left[\frac{F_\mu(\alpha_1, w, v_V)}{\alpha_1 \alpha_2} \right]$$

The condition of existence of nonzero solutions w' and α_1' reduces to a dispersion quadratic equation whose roots are of the form

$$2\omega^{(1,2)} = 2Vk + i f_w \pm \left\{ -f_w^2 - \frac{4\alpha_1\alpha_2}{\chi_*} \left(\frac{\chi}{\chi_*} \right) v_{12}^2 k^2 - i \frac{4\alpha_1\alpha_2}{\chi_*} f_\alpha k \right\}^{1/2} \tag{3.6}$$

Separating the imaginary part of $\omega^{(1,2)}$ that determines the exponent of D, we obtain

$$D^{(1,2)} = \mathrm{Im}\,\{\omega^{(1,2)}\} = \frac{f_w}{2} \pm \frac{1}{2\sqrt{2}} \left\{ f_w^2 + \frac{4\alpha_1\alpha_2}{\chi_*} \left(\frac{\chi}{\chi_*} \right) v_{12}^2 k^2 + \left(\left[f_w^2 + \frac{4\alpha_1\alpha_2}{\chi_*} \left(\frac{\chi}{\chi_*} \right) v_{12}^2 k^2 \right]^2 + \left[\frac{4\alpha_1\alpha_2}{\chi_*} f_\alpha k \right]^2 \right)^{1/2} \right\}^{1/2} \tag{3.7}$$

which shows that when $v_{12} \neq 0$, $\alpha_2 > 0$

$$D^{(2)} < 0,\ D^{(2)} \to -\infty \text{ when } k \to \infty \tag{3.8}$$

i.e. in the presence of relative motion of phased v_{12} in the initial state and nonzero concentration of the dispersed phase α_2 there exists, as shown in Sect.2, convection perturbations that disturb the stability of the steady homogeneous state (2.3).

As in the case considered in Sect.2, we conclude that the Cauchy problem formulation for Eqs.(3.1) with initial conditions close to the steady homogeneous state W_0 defined in (2.3) is incorrect when $v_{12} \neq 0$, $\alpha_2 > 0$ owing to the unbounded increase of the quantity $(-D)$ as $k \to \infty$

This conclusion is based on the behavior of system (1.1) or (3.1) when the wave length $\lambda = 2\pi k^{-1}$ approaches zero, while the indicated systems of equations correctly define the behavior of a disperse mixture only when the characteristic distances considered in the problem (in particular the wave length λ) are considerably greater than the dispersed particle dimension a. It is, therefore possible to assume that the boundless increase of the exponent of $(-D)$ as $\lambda \to 0$ is the consequence of the neglect in equations of type (1.1) or (3.1) of some dissipation processes that occur when ultrashort waves $(\lambda \leqslant a)$ pass through disperse media.

The allowance or nonallowance for phase compressibility, interphase forces of the inertial type (Archimedean force, apparent additional mass forces) does not affect the anomalous character of behavior of variation of D as $\lambda \to 0$. In connection with this it may be necessary, when solving equations of the type (1.1) or (3.1) as definitions of unsteady flows of disperse media, to suppress the unobtainable in reality ultrashort wave perturbation increase by introducing in these equations some additional terms.

The system of equations correctly defines the behavior of long-wave perturbations when $(\lambda \to \infty$ or $\lambda \ll a)$. Existence of convection perturbations when $v_{12} \neq 0$, $\alpha_2 > 0$ for which the exponent of D is negative (see (2.9)) also indicates physical instability of the homogeneous steady state solution considered. In particular, the steady homogeneous mode of dispersed particle sedimentation when $\rho_2^\circ > \rho_1^\circ$ or rise when $\rho_2^\circ < \rho_1^\circ$ is unstable owing to gravity.

If the interphase Archimedean force $\alpha_2 \partial p/\partial x$ and the apparent additional mass forces are neglected in equations of momenta of the phase system (3.1), it is then necessary to substitute in the characteristic equations (3.4), (3.6), and (3.7) zero for χ, and it is clear from (3.4) that the system has become hyperbolic, but (3.7) implies that it solution is, as previously, unstable. In a unique particular case in which $\chi = 0$ and $\partial K_\mu/\partial\alpha_1 = 0$ the solution is neutrally stable $\operatorname{Im}\{\omega^{(1,2)}\}=0$.) We point out that this case was virtually never considered in investigations of solutions of specific problems (*).

4. The rate of instability development depends on the wave number k and approaches infinity as the wave length $\lambda=2\pi k^{-1}$ approaches zero. Taking into account the validity of system (1.1) in the case when the problem characteristic distances (including wave length λ) greatly exceed the distance between particles, and assuming that perturbations with shorter waves would be damped owing to the dissipation processes not defined in the system of Eqs.(1.1), we obtain the condition for the maximum of k. It can be derived from the system of Eqs.(1.1) and is $k_{\max}\sim\alpha_2^{1/3}\pi/(an)$, where a is the radius of a particle, and the minimum length of a wave whose development can be analyzed using Eqs.(1.1) and (3.1) is equal n times the mean distance ($\lambda\sim na\alpha_2^{-1/3}$, $n\gg 1$) between particles. The obtained constraint enables us to evaluate the actual time of development of instability in gas-liquid streams. Note that when the problem is solved numerically using the system of Eqs.(1.1), perturbations with wave length of the order of the difference grid dimension $\lambda\sim\Delta x$ appear in the system. Then, if $\Delta x\lesssim k_{\max}^{-1}$, we may have numerical instability. In such cases it is necessary to provide for the suppression of such parasitic perturbations using, for instance, additional terms of pseudoviscosity type.

Let us estimate the rate of small perturbation increase under conditions characteristic for gas-liquid streams. We specify the viscous friction force in gas with liquid drops, as well as in water with gas bubbles, as in /1/, in conformity with (1.2) and (1.3) with $\varphi_f=(1+\alpha_2)/\alpha_2$. At the limit $\alpha_2\to 0$ the following relations hold:

$$f_w^2\gg 4\frac{\gamma}{\chi_*}\alpha_1\alpha_2 v_{12}^2k^2,\quad f_w^2\gg 4\frac{\alpha_1\alpha_2}{\chi_*}f_\alpha k \tag{4.1}$$

In the case of gas with liquid drops, taking into account that $\rho_1^\circ\ll\rho_2^\circ$ and using (4.1), from (3.7) with $\alpha_2>\rho_1^\circ/(6\rho_2^\circ)$ we can obtain

$$\operatorname{Im}\{\omega^{(2)}\}=-\frac{2\pi}{n^2}\frac{\rho_2^\circ v_{12}^2}{\mu_1}\alpha_2^{8/3} \tag{4.2}$$

For the characteristic time Δt of instability development we have

$$\Delta t=\frac{1}{|\operatorname{Im}\{\omega^{(2)}\}|}=\frac{9n^2}{\pi^2}\alpha_2^{-8/3}R_{12}^{-2}\left(\frac{\rho_1^\circ}{\rho_2^\circ}\right)^2\tau_\mu,\quad R_{12}=\frac{2\rho_1^\circ a\,|v_{12}|}{\mu_1},\quad \tau_\mu=\frac{2a^2\rho_2^\circ}{9\mu_1} \tag{4.3}$$

In gas with liquid drops flow mode R_{12} is usually not very large and $\alpha_2\ll 1$, hence the time of instability development is fairly long. For example, for parameters

$$p\sim 10^5\,\text{Pa},\quad \alpha_2\sim 10^{-2},\quad v_{12}\sim 10^{-1}\,\text{m/s},\quad a\sim 10^{-4}\,\text{m},\quad \mu_1\sim 10^{-5}\,\text{kg/(m.s)} \tag{4.4}$$

characteristic for flows of gas with liquid drops, from (3.10) with $n=20$ we obtain $\Delta t\sim 6$ s.

If the characteristic time Δt of instability development is considerably longer than characteristic time of the problem (the time particles remain under conditions (4.4)), the indicated instability may not develop in practice.

From (3.7), taking into account (4.1) and the condition that $\rho_1^\circ\gg\rho_2^\circ$, for the flow of liquid with bubbles we obtain

$$\operatorname{Im}\{\omega^{(2)}\}=-\frac{\pi^2}{3n^2}\frac{\rho_1^\circ v_{12}^2}{\mu_1}\alpha_2^{5/3} \tag{4.5}$$

For the characteristic time Δt of instability development we have

$$\Delta t=\frac{1}{|\operatorname{Im}\{\omega^{(2)}\}|}=\frac{54n^2}{\pi^2}\alpha_2^{-5/3}R_{12}^{-2}\left(\frac{\rho_1^\circ}{\rho_2^\circ}\right)\tau_\mu$$

Using the characteristic for bubble flow parameters

$$p\sim 10^5\,\text{Pa},\quad \alpha_2\sim 10^{-1},\quad v_{12}\sim 10^{-1}\,\text{m/s},\quad a\sim 10^{-3}\,\text{m},\ \mu_1\sim 0.3\cdot 10^{-3}\,\text{kg/m.s}$$

from (3.7) with $n=20$ we obtain $\Delta t\approx 0.5$ s.

If the bubble type of flow of a two-phase mixture the carrier medium velocity is not high (e.g., $v_1=0$ in a bubbling column), such instability can develop in real equipment and be the determining factor.

Note that the characteristic times Δt of instability development are proportional to $\alpha_2^{-8/3}$ and $\alpha_2^{-5/3}$ in the drop and bubble type of flow, respectively. The form of obtained estimates

*) The authors' attention to this exceptional case with neutral stability was drawn by A.N. Kraiko.

(4.2) and (4.5) differs from that of similar estimates in /4,5/ owing to the effect of viscous friction force F_μ which was not considered in /4,5/.

5. In a wide class of problems free of any noticeable effect of outer mass forces, for example in shock wave investigations, the finiteness of the phase slipping velocity v_{12} with the associated with it instability appear only in some time interval or in some zone, outside of which v_{12} approaches zero. Let us consider the stability of one of the unsteady state solutions (hence different from (2.3)) of the system of Eqs.(3.1) for a mixture of two incompressible phases in the absence of external mass forces $g = 0$.

If at the initial instant $t = 0$ the medium state is homogeneous, i.e. its parameters are independent of x, but unstable owing to the non-coincidence of phase velocities, then in the inertial coordinate system moving at instant $t = 0$ at the dispersed phase velocity, for the system of Eqs.(3.1) we have the following initial conditions:

$$t = 0, \quad v_1 = v_{10}, \quad v_2 = 0, \quad \alpha_1 = \alpha_{10}$$

As the boundary conditions we take the condition of constancy of the medium mean volumetric flow rate that can be obtained by moving a piston at constant velocity v_0

$$v_V = \alpha_1 v_1 + \alpha_2 v_2 = v_0 = \alpha_{10} v_{10}$$

Because of the carrier action, particles are set in motions, and the velocities of phases are gradually equalized.

Solution of this problem for $w^\circ(t)$ is of the form

$$w^\circ(t) = \frac{v_0}{3}\left[2\left(1 - \frac{\rho_2^\circ}{\rho_1^\circ}\right) : \left(1 + 2\frac{\rho_2^\circ}{\rho_1^\circ} + 3\frac{\alpha_{20}}{\alpha_{10}}\right)\right] \exp(-t/\tau_w) \tag{5.1}$$

$$\tau_w = \frac{1}{f_w} = \frac{\tau_\mu}{2}\frac{\rho_1^\circ}{\rho_2^\circ}\alpha_{10}\left(1 + \frac{2\rho_2^\circ}{\rho_1^\circ} + \frac{3\alpha_{20}}{\alpha_{10}}\right)$$

$$\alpha_1^\circ = \alpha_{10}, \quad \alpha_{20} = 1 - \alpha_{10}, \quad v_V = v_0/\alpha_{10}$$

It can be shown that as $t \to \infty$ the solution will indicate the disappearance of phase slipping ($v_{12} \to 0$). Let us investigate the stability of the derived solution by investigating perturbed solutions at fixed mean volumetric flow rate

$$w = w^\circ(t) + w'(x, t), \quad \alpha_1 = \alpha_1^\circ + \alpha'(x, t) \quad (w' \ll w^\circ, \alpha' \ll \alpha_1^\circ, v_V' = 0)$$

Then from (3.3) we obtain for w' and α' a linear system of equations whose coefficients V, v_{12}, f_α, f_w, unlike in Sect.3, are not constants but functions of time determined by the initial solution (5.1). Let us assume that perturbations w', α' are integrable in quadratures (belong to class L_2). Then the unknown functions have their Fourier tansforms W' and A'.

$$w'(x, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} W'(k, t)\, e^{ikx}\, dk, \qquad \alpha'(x, t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} A'(k, t)\, e^{ikx}\, dk \tag{5.2}$$

As the result of transformations, the system of equations for W' and A' reduces to degenerate hypergeometric equations. Finally, it is possible to obtain for W' and A', as $t \to \infty$, an estimate for the infinitely short waves which are of the greatest interest

$$\lim A' \sim D_0(k)e^{hk} \tag{5.3}$$

$$\lim W' \sim D_0'(k)e^{hk}, \quad h > 0, \quad t \to \infty, \quad k \to \infty$$

since their rate of growth is the highest. In these expressions $D_0(k)$ is a linear combination of Fourier transforms $A'(k, t)$ and $W'(k, t)$ at the initial instant of time $t = 0$.

It follows from (5.3) that the formulation of the Cauchy problem is in this case incorrect, and the derived homogeneous unsteady solution (5.2) is unstable. Nevertheless, as in the class of functions whose Fourier harmonics approach zero as $k \to \infty$, more rapidly than e^{-hk}, the Cauchy problem is "conditionally correct" /9,10/. The necessary condition for satisfying this constraint is the infinite differentiability of the imposed perturbation. Localized perturbations of the form $P_n(x)e^{-dx^2}$ (for any $d > 0$), where $P_n(x)$ is an arbitrary n-th power polynomial, satisfy that condition.

Note that the requirement for fairly rapid decrease of the amplitude of Fourier harmonics as $k \to \infty$ in the class of functions for which the "conditional correctness" of the Cauchy problem holds is related in Sect.3 to the question of inapplicability of system (1.1) for ultrashort perturbations.

A similar reasoning is valid for steady state solutions with constant slipping. In that case constraints on short-wave amplitude result in that in a finite time interval small perturbations remain small.

Thus the above analysis shows that, although system (3.1) (and (1.1)) has imaginary characteristics and, consequently, is nonhyperbolic, there exists a class of functions in which the Cauchy problem for system (3.1) (the system (1.1) is "conditionally correct".

6. Various additional mechanisms of flow stabilization, which are not accounted for in the systems of Eqs.(1.1) and (3.1), can exist in real flows. Consider the effect of internal pressure of inclusions, resulting from the interaction induced by the chaotic motion, of the stability of flow. For investigating this we introduce the pressure of a "pseudogas" of dispersed particles defined by the formula

$$p_2 = p_{20}\left(\frac{\alpha_2}{\alpha_{2*}-\alpha_2}\right)^n, \quad p_{20} > 0 \tag{6.1}$$

where $\alpha_{2*} \sim 0.5$ is the maximum possible particle concentration (close packing). The equation of momentum of particles in system (3.1) is then of the form

$$\rho_2{}^\circ\alpha_2 \frac{d_2 v_2}{dt} = -\alpha_2 \frac{\partial p}{\partial x} - \frac{\partial p_2}{\partial x} + \frac{\rho_1{}^\circ\alpha_1\alpha_2}{2}\left(\frac{d_1 v_1}{dt} - \frac{d_2 v_2}{dt}\right) + F_\mu + \rho_2{}^\circ\alpha_2 g \tag{6.2}$$

It can be shown that in that case formulas (3.3), (3.4), (3.6), and (3.7) remain valid when the quantities (χ/χ_*) are replaced by K

$$K = \frac{\chi}{\chi_*} - \frac{2np_{20}\alpha_{2*}\alpha_2^{n-2}}{(\alpha_{2*}-\alpha_2)^{n+1}\,3\rho_1{}^\circ v_{12}{}^2} \tag{6.3}$$

Formulas (3.4), (3.7), and (6.3) show that for $-1 < n < 2$ system (3.1) becomes hyperbolic and its solution stable as $\alpha_2 \to 0$ and $\alpha_2 \to 0.5$. There exists then a $p_{20}{}^*$, such that when $p_{20} > p_{20}{}^*$, system (3.1) is hyperbolic and its solutions are stable for any α_2. Indeed, for any fairly high p_{20} parameter K is negative, eigenvalues of (3.4) become real, and the damping decrement in (3.7) becomes positive.

The flow can evidently be stabilized also by other volume forces impeding the bunching of inclusions. For instance, it can be shown that the allowance for turbulence diffusion (introduction of terms $D\partial^2\alpha_i/\partial x^2$ and $D\rho_i{}^\circ\alpha_i\partial v_i/\partial x^2$, where D is the diffusion coefficient, in the right-hand sides of respective equations of continuity and momentum of system (3.1)) also stabilizes the flow.

7. We conclude by presenting a simple physical model illustrating the origin of instability in two-speed flows, and the effect stabilizing mechanisms. Distribution of particle velocities along the x axis at some instant of time is shown in Fig.1. It will be seen that in the time interval $\Delta t = \Delta x/\Delta v_2$ points A and B move, respectively, to points A'', B'', i.e. they prove to be at the same point of space. Such "whiplash-like motion" of particles results in an unbounded growth of α_2, and this is the cause of instability development. Such instability can obviously be eliminated by applying additional forces that would inhibit such sudden motion. For instance, the introduced above pressure p_2 of a pseudogas impedes the concentration of particles.

Another example of two-speed flow which does not result in a whiplash motion even in the absence of stresses in the second phase ($\sigma_2 = 0$) is shown schematically in Fig.2. Because of interaction between particles with the carrier phase, the mean velocities of particles A and B may become equal ($\Delta v' < \Delta v$) before the occurance of whiplash motion. The possibility of its occurrence is solely determined by the initial velocity distribution and particle concentration. With good initial distributions that do not result in whiplash, the small perturbations inserted in the flow remain small, as in the problem considered in Sect.5 on the drawing in of particles by the motion of the carrier medium.

When the volume concentration of particles is small and the dependence of the carrier medium velocity v_1 on the volume concentration α_2 can be neglected ($v_1 = \text{const}$), it can be assumed that particles A and B move independently. It is then possible to consider the case, when the carrier medium with constant velocity and equal to that of particle B, and obtain the following condition of whiplash absence for particles of mass m_2:

$$\frac{v_B - v_A}{\Delta x} \sim \frac{\partial v}{\partial x} < \frac{1}{\tau_\mu}, \quad \tau_\mu = \frac{\rho_2{}^\circ}{K_\mu}$$

Note that this condition does not take into account the effect of particle volume concentration on the velocity of the carrier medium.

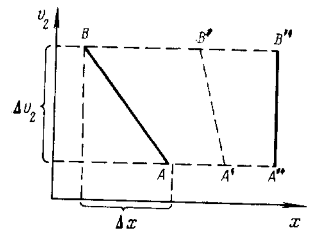

Fig.1

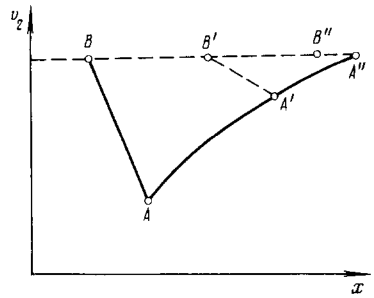

Fig.2

We shall describe two mechanisms by which the particle volume concentration α_2 affects the flow stability. Consider the flow of liquid in a vertical pipe with bubbles of gas fed at its lower end floating upward. We assume that initially the liquid velocity is zero and the volume concentration of bubbles along the pipe is constant. Analysis of the continuity equation shows that, if at some instant of time the particle concentration in section dx increases, the velocity of liquid in that section becomes negative. The appearance of negative velocity of the carrier phase lowers the pressure in section dx (in conformity with the equation of momentum conservation). The lower pressure region by attracting into it more bubbles will further increase bubble concentration in section dx. This mechanism was noted in /4/; it is defined in Eq.(3.10) by the term $Q = 4\chi\chi_*^{-2}v_{12}^2\alpha_1\alpha_2k^2$. Another term in that equation, which implies the possibility of unlimited growth of perturbations is $S = 4\alpha_1\alpha_2\chi_*^{-1}f_\alpha k$ which shows that at the limit, as $k \to \infty$, the most significant term is Q. However, if the condition imposed by the applicability of system (1.1) on the minimal wave length $k_{max} \leqslant \alpha_2^{1/3}\pi(an)^{-1}$ (Sect.4) is taken into account, the question which of the terms defines the most rapidly increasing perturbations cannot be answered a priori. It was shown in Sect.4 on two examples that in the case of gas with liquid drops type of flow the most rapid perturbation growth corresponds to the S term, while in that of bubble type flow it is the Q term. This explains the difference in exponents of α_2 in (4.2) and (4.5).

The authors thank A.A. Barmin, A.N. Kraiko, and A.G. Kulikovskii for discussions and valuable advice.

REFERENCES

1. NIGMATULIN R.I., Fundamentals of Mechanics of Heterogeneous Media. Moscow, NAUKA, 1978.
2. RAKHMATULIN Kh.A., Fundamentals of the gas dynamics of interpenetrating motions of compressible media. PMM, Vol.20, NO.2, 1950.
3. KRAIKO A.N. and STERNIN L.E., Theory of flows of a two-velocity continuous medium containing solid or liquid particles. PMM, Vol.29, No.3, 1965.
4. IORDANSKII S.V. and KULIKOVSKII A.G., On the motion of fluid containing small particles. Izv. Akad. Nauk SSSR, MZhG, No. 4, 1977.
5. IORDANSKII S.V. and KULIKOVSKII A.G., Correction to the paper by Iordanskii S.V. and Kulikovskii A.G., On the motion of fluid containing small particles. Izv. Akad. Nauk SSSR, MZhG, No.4, 1978.
6. OVSIANNIKOV L.V., Models of two-layer "shallow water". PMTF, No.2, 1979.
7. BANERJEE S. and HANCOX W.T., Transient Thermohydraulics Analysis for Nuclear Reactors. Proc. 6-th Internat. Heat Transfer Conference, Toronto, 1978. Ottawa, 1978.
8. IVANDAEV A.I. and NIGMATULIN R.I., Singularities of weak perturbation propagation in two-phase media with phase transitions. PMTF, No.5, 1970.
9. LAVRENT'EV M.M., On the Cauchy problem for the Laplace equation. Izv. Akad. Nauk, SSSR, Ser. Matem., Vol.20, No.6, 1956.
10. GODUNOV S.K., Equations of Mathematical Physics. Moscow, NAUKA, 1979.

Translated by J.J.D.